MEM6810 Engineering Systems Modeling and Simulation
工程系统建模与仿真

Theory Analysis

Lecture 2: Elements of Probability and Statistics

SHEN Haihui 沈海辉

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

shenhaihui.github.io/teaching/mem6810f
shenhaihui@sjtu.edu.cn

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上海交通大學 SHANGHAI JIAO TONG UNIVERSITY | 董浩云智能制造与服务管理研究院 CY TUNG Institute of Intelligent Manufacturing and Service Management (中美物流研究院) (Sino-US Global Logistics Institute)

Contents

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  3. Countably additive: If $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, is a **countable** sequence of **disjoint** sets, then $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

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- Example 1: Flip a fair coin.
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- Example 3: Randomly "draw" a number in $[0, 1]$.
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  - A more practical and interesting $\mathcal{F}$ is the one that contains all intervals (no matter open or closed) on $[0, 1]$.

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- Clearly, mutual independence implies pairwise independence, but not vice versa!

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- Remark: For event A, if $\mathbb{P}(A) = 1$, then we say A happens **almost surely** (a.s.).

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  - A popular convention is to denote the RVs by upper-case letters (e.g., X and Y) and their realizations by lower-case letters (e.g., x and y).

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  - Under $(\Omega, \mathcal{F}, \mathbb{P})$, let $X(\omega) = \omega$ for $\omega \in [0, 1]$.

- The **cumulative distribution function** (CDF) of a RV X, denoted by $F : \mathbb{R} \to [0, 1]$, is defined by

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  - $F(x)$ is right-continuous, that is, for any $x_0 \in \mathbb{R}$,

$$\lim_{x\downarrow x_0} F(x) = F(x_0).$$

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- It is easy to see that $F(x) = \sum_{y \in (-\infty, x]} p(y)$.

- A RV X is said to be **continuous** if there exists a **probability density function** (pdf) $f(x)$ such that

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- Observe that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

- The **joint** CDF of RVs X and Y, denoted by $F : \mathbb{R}\times\mathbb{R} \to [0,1]$, is defined by

$$\begin{aligned} F(x,y) &:= \mathbb{P}(X \le x, Y \le y) \\ &= \mathbb{P}(\{\omega : X(\omega) \le x\} \cap \{\omega : Y(\omega) \le y\}), \ \forall x, y \in \mathbb{R}. \end{aligned}$$

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- Observe that $\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$.

- Given the random vector $(X, Y)^\mathsf{T}$, the distribution of X or Y is called the **marginal distribution**.
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- For Y, its marginal CDF, and pmf or pdf, can be determined similarly.

Univariate Transformation - Continuous Case

Let X be a continuous RV, and $Y = g(X)$, where g is a **monotone** function. Let

$\mathcal{X} := \{x : f_X(x) > 0\}$ and $\mathcal{Y} := \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$.

Suppose that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$. Then,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right|, & y \in \mathcal{Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Bivariate Transformation - Continuous Case

Let $(X, Y)^\intercal$ be a continuous bivariate random vector, and $U = g_1(X, Y)$ and $V = g_2(X, Y)$. Let

$$\mathcal{A} := \{(x, y) : f_{X,Y}(x, y) > 0\},$$
$$\mathcal{B} := \{(u, v) : u = g_1(x, y), v = g_2(x, y) \text{ for some } (x, y) \in \mathcal{A}\}.$$

Suppose that $u = g_1(x, y)$ and $v = g_2(x, y)$ define a **one-to-one** transformation of $\mathcal{A}$ **onto** $\mathcal{B}$, and $x = h_1(u, v)$ and $y = h_2(u, v)$ have continuous partial derivatives on $\mathcal{B}$. Then,

$$f_{U,V}(u, v) = \begin{cases} f_{X,Y}(h_1(u, v), h_2(u, v)) \, |J| \, , & (u, v) \in \mathcal{B}, \\ 0, & \text{otherwise,} \end{cases}$$

given that J is not identically 0 on $\mathcal{B}$, where J is the Jacobian

Bivariate Transformation - Continuous Case (Cont'd)

of the transformation, i.e.,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v},$$

and

$$\frac{\partial x}{\partial u} = \frac{\partial h_1(u,v)}{\partial u}, \quad \frac{\partial x}{\partial v} = \frac{\partial h_1(u,v)}{\partial v},$$
$$\frac{\partial y}{\partial u} = \frac{\partial h_2(u,v)}{\partial u}, \quad \frac{\partial y}{\partial v} = \frac{\partial h_2(u,v)}{\partial v}.$$

- If $(X, Y)^\intercal$ is discrete, for any y such that $\mathbb{P}(Y = y) = p_Y(y) > 0$, the **conditional** pmf of X given that $Y = y$ is defined as

$$p(x|y) \coloneqq \mathbb{P}(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)}.$$

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&= \frac{\int_{-\infty}^{x} f(t,y)\mathrm{d}t}{f_Y(y)}.
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2. Then, $f(x|y) = \frac{\partial}{\partial x}F(x|Y=y) = \frac{\frac{\partial}{\partial x}\int_{-\infty}^{x} f(t,y)\mathrm{d}t}{f_Y(y)} = \frac{f(x,y)}{f_Y(y)}$.

- Two RVs X and Y are said to be statistically **independent**, which can be denoted as $X \perp Y$, when, for any $x, y \in \mathbb{R}$,

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- RVs $X_1, \ldots, X_n$ are pairwise independent if for any $i \neq j$, $X_i \perp X_j$.

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$$\mathbb{E}[X] := \int_\Omega X(\omega) \mathrm{d}\,\mathbb{P}(\omega),$$
provided that $\int_\Omega |X(\omega)| \mathrm{d}\,\mathbb{P}(\omega) < \infty$ or $X \geq 0$ a.s., where the integral is the Lebesgue integral, rather than the Riemann integral.

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  - **Covariance**: $\mathrm{Cov}(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$.

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- In general, $X \perp Y \;\; \overset{\Longrightarrow}{\not\Longleftarrow} \;\; \rho(X, Y) = 0 \iff \operatorname{Cov}(X, Y) = 0$.

- For integer n, $\mathbb{E}[X^n]$ is called the nth **moment** of X, and $\mathbb{E}[(X-\mathbb{E}[X])^n]$ is called the nth **central moment** of X.
- Some special moments:
  - Mean (1st moment): $\mu := \mathbb{E}[X]$.
  - **Variance** (2nd central moment):
    $\sigma^2 := \mathrm{Var}(X) := \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.
- Linear association:
  - **Covariance**:
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  - **Correlation**: $\rho(X,Y) := \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$.
- In general, $X \perp Y \quad \underset{\not\Leftarrow}{\Rightarrow} \quad \rho(X,Y) = 0 \iff \mathrm{Cov}(X,Y) = 0$.
- If $(X,Y)^\intercal$ follows a bivariate normal distribution,[†] then $X \perp Y \iff \rho(X,Y) = 0$.

[†] **CAUTION:** It means MORE than that X and Y both follow a normal distribution! More details latter.

- The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|y] := \begin{cases} \sum_{x\in\mathbb{R}} xp(x|y), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} xf(x|y)\mathrm{d}x, & \text{if } X \text{ is continuous.} \end{cases}$$

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- If $X \perp Y$, then $\mathbb{E}[X|y] = \mathbb{E}[X|Y] = \mathbb{E}[X]$, and $\mathrm{Var}(X|y) = \mathrm{Var}(X|Y) = \mathrm{Var}(X)$.

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- If $M_X(t)$ is finite for t in some neighborhood of 0 (i.e., there is an $h > 0$ such that for all $t \in (-h, h)$, $M_X(t) < \infty$), then,

$$\mathbb{E}[X^n] = \frac{\mathrm{d}^n}{\mathrm{d}t^n} M_X(t)\bigg|_{t=0}, \; n \in \mathbb{N}.$$

- $X \sim \mathrm{Bernoulli}(p)$ or $\mathrm{Ber}(p)$, if

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  - Given $X_1 + X_2 = n$, $X_1 \sim \text{B}(n, \lambda_1/(\lambda_1 + \lambda_2))$.

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  - $F(x) = 1 - e^{-\lambda x}$, $\mathbb{P}(X > x) = 1 - F(x) = e^{-\lambda x}$.
  - $\mathbb{E}[X] = 1/\lambda$, $\mathrm{Var}(X) = 1/\lambda^2$.

- $X \sim \mathrm{uniform}(a,b)$ or $\mathrm{Unif}(a,b)$ with $a < b$, if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

  - $\mathbb{E}[X] = \frac{b+a}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$.

- $X \sim \mathrm{exponential}(\lambda)$ or $\mathrm{Exp}(\lambda)$, with $\lambda > 0$, if its pdf is given by

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  - **Memoryless Property**: For $s > t \geq 0$,

$$\mathbb{P}(X > s | X > t) = \frac{\mathbb{P}(X > s, X > t)}{\mathbb{P}(X > t)} = \frac{\mathbb{P}(X > s)}{\mathbb{P}(X > t)} = \frac{e^{-\lambda s}}{e^{-\lambda t}} = e^{-\lambda(s-t)}$$
$$= \mathbb{P}(X > s - t).$$

- If $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ are independent, then $\min\{X_1, X_2\} \sim \text{Exp}(\lambda_1 + \lambda_2)$.

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- $X \sim \text{Gamma}(\alpha, \lambda)$ in shape & rate parametrization with $\alpha, \lambda > 0$, if its pdf is given by

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  - $\alpha = p/2$, where p is an integer, and $\lambda = 1/2 \Longrightarrow$ chi-square distribution with p degrees of freedom, denoted as χ^2_p.

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  - $\alpha = 1, \beta > 1 \implies$ strictly decreasing
  - $\alpha < 1, \beta < 1 \implies$ U-shaped
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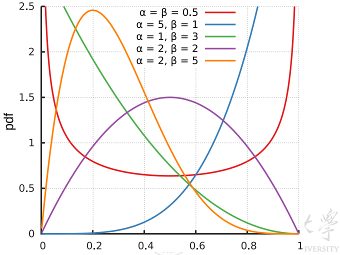

- $X \sim$ Student's t distribution with p degrees of freedom, denoted as t_p, where p is an integer, if its pdf is given by

$$f(x) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + x^2/p)^{(p+1)/2}}, \quad x \in \mathbb{R}.$$

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  - Z is also known as the **standard normal** RV.
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If $Y \sim \chi_1^2$, i.e., $Y \sim \mathrm{Gamma}(1/2, 1/2)$, it means its pdf is

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$$\begin{aligned} f(y) &= \frac{\mathrm{d}}{\mathrm{d}y}F(y) = \phi(\sqrt{y})\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} - \phi(-\sqrt{y})\frac{\mathrm{d}}{\mathrm{d}y}(-\sqrt{y}) \\ &= 2\phi(\sqrt{y})\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} = \frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}}y^{-\frac{1}{2}}. \end{aligned}$$

If $Y \sim \chi^2_1$, i.e., $Y \sim \mathrm{Gamma}(1/2, 1/2)$, it means its pdf is

$$f(y) = \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})}y^{-\frac{1}{2}}e^{-\frac{y}{2}}.$$

The proof is completed by showing that $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}\mathrm{d}t = \sqrt{\pi}$, which can be seen if we convert to polar coordinates. ∎

- If $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi_p^2$ are independent, then $\frac{Z}{\sqrt{V/p}} \sim t_p$.

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Proof. Since $V \sim \chi_p^2$, by definition, its pdf is

$$f_V(v) = \frac{\left(\frac{1}{2}\right)^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} v^{\frac{p}{2}-1} e^{-\frac{1}{2}v}, \quad v \in (0, \infty).$$

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Let $T := \frac{Z}{\sqrt{V/p}} = \frac{Z}{Y}$. For $t \in \mathbb{R}$,

$$\mathbb{P}(T \le t) = \mathbb{P}\left(\frac{Z}{Y} \le t\right) = \mathbb{P}(Z \le tY) = \int_0^\infty \mathbb{P}(Z \le ty) f_Y(y)\mathrm{d}y. \quad \text{(Why?)}$$

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Then,

$$f_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(T \le t) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(Z \le ty) f_Y(y)\mathrm{d}y.$$

Proof. (Cont'd) Note that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(Z \leq ty) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{ty}\phi(z)\mathrm{d}z = y\phi(ty)$.

Proof. (Cont'd) Note that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(Z \le ty) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{ty}\phi(z)\mathrm{d}z = y\phi(ty)$. So,

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f_T(t) &= \int_0^\infty y\phi(ty) f_Y(y)\mathrm{d}y = \int_0^\infty y\phi(ty) 2py f_V(py^2)\mathrm{d}y \\
&= \int_0^\infty 2py^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2y^2}{2}} \cdot \frac{\left(\frac{1}{2}\right)^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} (py^2)^{\frac{p}{2}-1} e^{-\frac{1}{2}py^2}\mathrm{d}y
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Let $x := y^2$. Then, integration by substitution shows that

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\int_0^\infty y^p e^{-\frac{1}{2}(t^2+p)y^2}\mathrm{d}y = \frac{1}{2}\int_0^\infty x^{\frac{p-1}{2}}e^{-\frac{1}{2}(t^2+p)x}\mathrm{d}x =: \frac{1}{2}\int_0^\infty x^{\alpha-1}e^{-\lambda x}\mathrm{d}x,
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where $\alpha := \frac{p+1}{2}$ and $\lambda := \frac{1}{2}(t^2+p)$.

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where $\alpha \coloneqq \frac{p+1}{2}$ and $\lambda \coloneqq \frac{1}{2}(t^2+p)$. Recalling the pdf of $\Gamma(\alpha,\lambda)$, it is easy to see that $\int_0^\infty x^{\alpha-1}e^{-\lambda x}\mathrm{d}x = \Gamma(\alpha)/\lambda^\alpha$.

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\begin{aligned}
f_T(t) &= \frac{1}{\Gamma(\frac{p}{2})}\frac{1}{(p\pi)^{1/2}}2^{\frac{1-p}{2}}p^{\frac{p+1}{2}} \cdot \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})}{(1/2)^{(p+1)/2}(t^2+p)^{(p+1)/2}} \\
&= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{1}{(p\pi)^{1/2}}\frac{1}{(1+t^2/p)^{(p+1)/2}}.
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  - $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $i = 1, \ldots, k$.

- If $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is k dimensional, then
  - $\boldsymbol{Z} := \boldsymbol{A}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$, where $\boldsymbol{A}$ satisfies $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}^\intercal$ (Cholesky decomposition), $\boldsymbol{0} \in \mathbb{R}^k$, and $\boldsymbol{I} \in \mathbb{R}^{k \times k}$ denotes the identity matrix.

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  - $\boldsymbol{a} + \boldsymbol{B}\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{a} + \boldsymbol{B}\boldsymbol{\mu}, \boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}^\intercal)$.†

† The multivariate normal distribution will be degenerate if $\boldsymbol{B}$ does not have full row rank ($\boldsymbol{B}$ 不行满秩).

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  There exist $\boldsymbol{\mu} \in \mathbb{R}^k$ and $\boldsymbol{A} \in \mathbb{R}^{k\times\ell}$ such that $\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Z}$, where $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ with $\boldsymbol{0} \in \mathbb{R}^\ell$ and $\boldsymbol{I} \in \mathbb{R}^{\ell\times\ell}$.

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- Bivariate normal distribution: $(X_1, X_2)^\mathsf{T} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)^\mathsf{T}$, and

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and the joint pdf is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}.$$

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$$\begin{aligned} f(x_1, x_2) = {} & \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ & \times e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}. \end{aligned}$$

- To see $\rho = 0 \Longrightarrow X_1 \perp X_2$, let $\rho = 0$, and note

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} = f_{X_1}(x_1) f_{X_2}(x_2). \end{aligned}$$

- If $(X_1, X_2)^\intercal \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.

- If $(X_1, X_2)^\mathsf{T} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.

Proof. Note that

$$\boldsymbol{Y} := \begin{bmatrix} X_1 + X_2 \\ X_1 - X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} =: \boldsymbol{B} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

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$$\begin{aligned} \mathrm{Cov}(X_1 + X_2, X_1 - X_2) &= \mathrm{Cov}(X_1, X_1) - \mathrm{Cov}(X_2, X_2) \\ &= \sigma^2 - \sigma^2 = 0. \end{aligned}$$ ■

- There are many other relationships among various probability distributions.
  - See, for example, Song (2005);
  - Or, Leemis & McQueston (2008) and their online interactive graph `http://www.math.wm.edu/~leemis/chart/UDR/UDR.html`

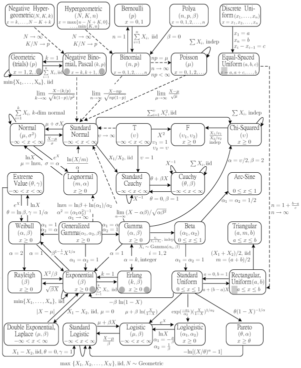

Figure: Relationships Among 35 Distributions (*from* Song (2005))

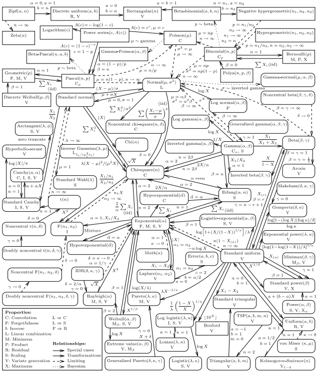

Figure: Relationships Among 76 Distributions (*from* Leemis & McQueston (2008))

Markov's Inequality

Let X be a RV. If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X = 0) < 1$, then, for any $r > 0$,

$$\mathbb{P}(X \geq r) \leq \frac{\mathbb{E}[X]}{r},$$

with equality if and only if

$$X = \begin{cases} r, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

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- Markov's Inequality has many variations, which are usually called Chebyshev's Inequality.

Chebyshev's Inequality

Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

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Chebyshev's Inequality

Let X be a RV. Then, for any $r, p > 0$,

$$\mathbb{P}(|X| \geq r) \leq \frac{\mathbb{E}[|X|^p]}{r^p},$$

$$\mathbb{P}(|X - \mu| \geq r) \leq \frac{\sigma^2}{r^2},$$

where $\mu := \mathbb{E}[X]$, and $\sigma^2 := \mathrm{Var}(X)$.

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- If $Z \sim \mathcal{N}(0,1)$, a tighter bound is available: For any $t > 0$,

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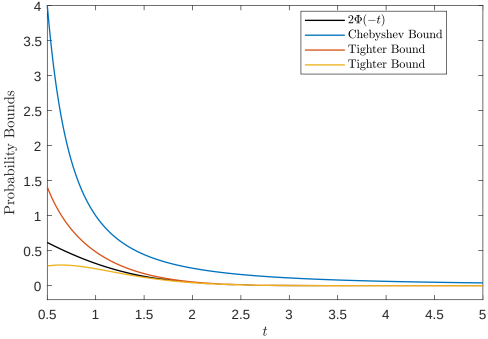

- A function $g(x)$ is **convex** if

$$g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda) g(y),$$

for all x and y, and $\lambda \in (0,1)$.

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Jensen's Inequality

Let X be a RV. If $g(x)$ is a convex function, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]),$$

with equality if and only if $g(x)$ is a linear function on some set A such that $\mathbb{P}(X \in A) = 1$.

Hölder's Inequality

Let X and Y be any two RVs, and let p and q be any two positive numbers (necessarily greater than 1) satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then,

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \{\mathbb{E}[|X|^p]\}^{1/p}\{\mathbb{E}[|Y|^q]\}^{1/q}.$$

Cauchy-Schwarz Inequality ($p = q = 2$)

Let X and Y be any two RVs, then

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Liapounov's Inequality ($Y \equiv 1$)

Let X be a RV, then for any $s > r > 1$,

$$\{\mathbb{E}[|X|^r]\}^{1/r} \le \{\mathbb{E}[|X|^s]\}^{1/s}.$$

Minkowski's Inequality

Let X and Y be any two RVs. Then, for $p \geq 1$,

$$\{\mathbb{E}[|X+Y|^p]\}^{1/p} \leq \{\mathbb{E}[|X|^p]\}^{1/p} + \{\mathbb{E}[|Y|^p]\}^{1/p}.$$

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- **Remark**: The preceding Hölder's Inequality (including its special cases) and Minkowski's Inequality also apply to numerical sums where there is no explicit reference to an expectation.

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- **Convergence Almost Surely** (a.s.), $X_n \xrightarrow{a.s.} X$:

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

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- **Convergence in Distribution**, $X_n \xrightarrow{d} X$, $X_n \Rightarrow X$, or $X_n \xrightarrow{d}$ distribution of X:
$$\lim_{n\to\infty} F_n(x) = F(x), \text{ for any continuous point } x \text{ of } F(x),$$
where F_n and F are CDF of X_n and X, respectively.

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- **Convergence in L^r Norm** ($r \in [1, \infty)$), $X_n \xrightarrow{L^r} X$:
$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0,$$
given $\mathbb{E}[|X_n|^r] < \infty$ for any $n \geq 1$ and $\mathbb{E}[|X|^r] < \infty$.

- Simple relationships:

$$\begin{array}{ccccc} X_n \xrightarrow{a.s.} X & \Longrightarrow & X_n \xrightarrow{p} X & \Longrightarrow & X_n \xrightarrow{d} X \\ & & \Uparrow & & \\ X_n \xrightarrow{L^s} X & \overset{s>r\geq 1}{\Longrightarrow} & X_n \xrightarrow{L^r} X & \Longrightarrow & \mathbb{E}[|X_n|^r] \to \mathbb{E}[|X|^r] \end{array}$$

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- Question: If $X_n \xrightarrow{d} X$ or $X_n \xrightarrow{p} X$ or $X_n \xrightarrow{a.s.} X$, does it imply $\mathbb{E}[X_n] \to \mathbb{E}[X]$?

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Monotone Convergence Theorem (MCT)

Suppose $X_n \xrightarrow{a.s.} X$, and $0 \le X_1 \le X_2 \le \cdots$ a.s.. Then $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

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Fatou's Lemma

Suppose $X_n \ge Y$ a.s. for all n where $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[\liminf_{n\to\infty} X_n] \le \liminf_{n\to\infty} \mathbb{E}[X_n]$. In particular, if $X_n \ge 0$ a.s. for all n, then the result holds.

Dominated Convergence Theorem (DCT)

Suppose $X_n \xrightarrow{a.s.} X$, $|X_n| \le Y$ a.s. for all n, and $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

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- An **even more general** result:
  Suppose $X_n \xrightarrow{p} X$, $|X_n| \le Y$ a.s. for all n, and $\mathbb{E}[|Y|^r] < \infty$ with $r \ge 1$. Then, $\mathbb{E}[|X_n|^r] < \infty$, $\mathbb{E}[|X|^r] < \infty$, and $X_n \xrightarrow{L^r} X$.

- $X = Y$ a.s., if *any one* of the following holds:
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- $X = Y$ a.s., if *any one* of the following holds:
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  $\Longrightarrow aX_n + bY_n \xrightarrow{d} aX + bc$; $X_nY_n \xrightarrow{d} cX$. (Due to CMT; also known as Slutsky's theorem)

Continuous Mapping Theorem (CMT)

Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X. Suppose g is a function that has the set of discontinuity points D such that $\mathbb{P}(X \in D) = 0$. Then,

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X_n \xrightarrow{a.s.} X \quad &\Longrightarrow \quad g(X_n) \xrightarrow{a.s.} g(X); \\
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- CMT also holds for **random vectors**.
- Caution: For convergence in L^r norm, stronger assumption of g than continuity is required to ensure $g(X_n) \xrightarrow{L^r} g(X)$.

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Properties of a Random Sample

- Let $X_1, \ldots, X_n$ be a **random sample** from a distribution with mean μ and variance σ^2, i.e., $X_1, \ldots, X_n$ are iid, and $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, $i = 1, \ldots, n$.

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Weak Law of Large Numbers (WLLN)

Suppose $X_1, \ldots, X_n$ are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{p} \mu$, as $n \to \infty$.

[†] Mutual independence can be weakened to pairwise independence; $\sigma^2 < \infty$ can be weakened to $\mathbb{E}[|X_i|] < \infty$.

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Suppose $X_1, \ldots, X_n$ are iid with mean μ and variance $\sigma^2 < \infty$.† Then, $\bar{X}_n \xrightarrow{a.s.} \mu$, as $n \to \infty$.

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- Note that for **normal** distribution, $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$, regardless of the value of n.

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Central Limit Theorem (CLT)

Suppose $X_1, \ldots, X_n$ are iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Then, as $n \to \infty$,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1).$$